The idea of Atiyah was to look at it as at an evaluation of a class $[E]$ with a class $[D]$ of the Dirac operator in some theory "dual to K-theory" $K_*(X)$.

Such a dual theory does exist and is called K-homology, stressing the cohomological nature of K-theory. Both theories are particular cases of a bivariant theory, which is the equivariant Kasparov theory $KK_*^G(-,-)$.

This will provide a category enriched in $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups, such that $K_G(-)$ becomes an enriched functor, i.e. there is a composition product

$$KK_*^G(C,B) \otimes KK_*^G(B,A) \longrightarrow KK_*^G(C,A),$$

and a natural map

$$KK_*^G(B,A) \longrightarrow \mathbf{Ab}^{\mathbb{Z}/2\mathbb{Z}}\left(K_*^G(B), K_*^G(A)\right).$$

What are desired properties of equivariant KK-theory?

Eleven desired properties of $KK_*^G(-,-)$:

0) First of all, every G-equivariant C^*-algebra map $B \to A$ should give an element in $KK_0^G(B,A)$, while every extension $A \rightarrowtail E \twoheadrightarrow B$ defines an element in $KK_1^G(B,A)$.

1) Bott periodicity, Thom isomorphism

2) Short sequence of non-unital G-C^*-algebra maps

$$0 \to I \to E \to Q \to 0$$

exact in **Vect** should induce a long exact sequence of $KK^G_*(B,-)$, $KK^G_*(-,A)$.

(by Bott periodicity: six term exact sequence)

3) twisting by a vector bundle (module structure over $K^*_G(X)$)

4) index theory $K^*_G(X) \to R(G)$ of elliptic G-equivariant differential operators.

The next property of KK-theory involves the so called minimal tensor product of C^*-algebras.

Definition. Let $\rho_i : A_i \to B(H_i)$, $i = 1, 2$ be representations of C^*-star algebras A_i.
The minimal C^*-norm on $A_1 \otimes A_2$ is defined as

$$\| \Sigma a_1 \otimes a_2 \|_{min} = \sup_{\rho_1, \rho_2} \| \Sigma \rho_1(a_1) \otimes \rho_2(a_2) \|$$

The completion of $A \otimes B$ with this norm is called the minimal (or spatial) tensor product of A_1, A_2.

Exercise 33. Show that the minimal tensor product of C^*-algebras is a C^*-algebra.

Remark. If $A_i = C(X_i)$, $i=1,2$, for X_i being compact Hausdorff spaces, then

$$C(X_1) \otimes C(X_2) = C(X_1 \times X_2).$$

This explains why the minimal tensor product is also called <u>spatial</u>: for noncommutative C^*-algebras A_i, $i=1,2$ this serves as the Cartesian product of noncommutative compact Hausdorff spaces.

Below $\otimes$ is a minimal tensor product of C^*-algebras.

5) $\forall A, B, C$ G-C^*-algebras there is an <u>external product</u>

$$KK_*^G(C,B) \longrightarrow KK_*^G(C\otimes A, B\otimes A).$$

6) <u>C^*-Morita invariance</u>: $A \rightsquigarrow \mathcal{K}(H)\otimes A$ and $B \rightsquigarrow K(H)\otimes A$ doesn't change KK_*;

where $K(H) \subset B(H)$, H separable, is an ideal consisting of the operators $T = \sum_{n=1}^{\infty} \lambda_n \langle \Psi_n, - \rangle \Phi_n$ where $(\Psi_n), (\Phi_n)$ o.n. sets of vectors, $0 < \lambda_n \xrightarrow[n\to\infty]{} 0$.

Exercise 34. Prove that $\mathcal{K}(H) \subset B(H)$ is a norm closed two-sided ideal.

7) $KK_*^G(\mathbb{C}, A) = K_G^*(A)$, (in particular $KK_*^G(\mathbb{C}, \mathbb{C}) = R(G)$)

8) Elliptic operators D (i.e. the Dirac operator) define elements $[D] \in KK_*^G(C(X), \mathbb{C})$.

9)
$$\begin{array}{ccccc} KK_*^G(\mathbb{C}, C(X)) & \otimes & KK_*^G(C(X), \mathbb{C}) & \longrightarrow & KK_*^G(\mathbb{C}, \mathbb{C}) \\ \| & & \| & & \| \\ K_*^G(C(X)) & \otimes & K_G^*(C(X)) & \longrightarrow & R(G) \\ [E] & \otimes & [D] & \longmapsto & \mathrm{ind}_G(D_E). \end{array}$$

Once $K_*^G(C(X))$ is contravariant in X (like cohomology)
$K_G^*(C(X))$ is covariant in X (like homology),
$KK_*^G(-,-)$ is called *bivariant*, and $K_G^*(-)$
is called (G-equivariant) *K-homology* of C^*-algebras
(or "noncommutative spaces" corresponding to C^*-algebras)

10) **Descent homomorphism.** For A, B G-C^*-algebras for G topological, then $KK^G_*(B,A)$ comes with a natural map

$$KK^G_*(B,A) \longrightarrow KK_*(G \ltimes B, G \ltimes A)$$

from G-equivariant KK-theory to KK-theory of <u>crossed product algebras</u>.

Example. G compact, $A = B = \mathbb{C}$. Then $KK^G_*(\mathbb{C},\mathbb{C}) = R(G)$ and the diagram

$$\begin{tikzcd}
KK_0(C^*G, C^*G) \arrow[r, "\cong"] & \mathbf{Ab}(R(G), R(G)) \\
KK_0^G(\mathbb{C}, \mathbb{C}) \arrow[u, "\text{descent}"] \arrow[r, "\cong"] & \mathbf{Mod}_{R(G)}(R(G), R(G)) \arrow[u, hook, "\not\cong"']
\end{tikzcd}$$

commutes, hence descent can be far from being an isomorphism.

Example. KK-theory can be used to prove the Connes–Thom isomorphism.

Before, we have invoke the following fact.

Fact. If G is a loc. compact group then for every C^*-algebra A

$$G \ltimes (C_0(G) \otimes A) \cong \mathcal{K}(L^2(G)) \otimes A \qquad \blacksquare$$

$$\begin{gathered} \cong \\ (G \ltimes C_0(G)) \otimes A \\ \cong \\ \mathcal{K}(L^2(G)) \otimes A. \end{gathered}$$

$$G \ltimes C_0(G) \hookrightarrow B(L^2(G))$$

left translation multiplication

norm closure of the image $\cong \mathcal{K}(L^2(G))$, the only norm closed proper ideal in $B(L^2(G))$.

Theorem. (Connes–Thom isomorphism)

$$K_*(\mathbb{R} \ltimes A) \cong K_{*+1}(A).$$

Proof. [Sketch of] By Bott periodicity

$$K^*(\mathbb{R}) \cong K^{*-1}(pt) \cong K^{*+1}(pt).$$

This can be proved by constructing $D \in KK_1(C_0(\mathbb{R}), \mathbb{C})$ and $\eta \in KK_1(\mathbb{C}, C_0(\mathbb{R}))$ which are inverse to each other.

Note that $\mathbb{R}$ acts on itself by translations and trivially on $\mathbb{C}$.

then D and η can be taken equivariant:

$$D \in KK_1^{\mathbb{R}}(C_0(\mathbb{R}), \mathbb{C}), \quad \eta \in KK_1^{\mathbb{R}}(\mathbb{C}, C_0(\mathbb{R})).$$

still mutually inverse in $KK_*^{\mathbb{R}}$.

$D = i\frac{d}{dx}$ on $\mathbb{R}$ is a Dirac operator

(note D is translation invariant),

η comes from the C^*-algebra extension

$$C_0(\mathbb{R}) \rightarrowtail C_0((-\infty, \infty])) \twoheadrightarrow \mathbb{C}.$$

Now use an exterior product: for G-C^*-algebra A

$$KK_1^{\mathbb{R}}(C_0(\mathbb{R}),\mathbb{C}) \xrightarrow{\mathrm{id}_A} KK_1^{\mathbb{R}}(C_0(\mathbb{R})\otimes A, \mathbb{C}\otimes A) .$$

$\Rightarrow$ $\forall$ $\mathbb{R}$-C^*-algebra A there is an invertible element in $KK_1^{\mathbb{R}}(C_0(\mathbb{R})\otimes A, \mathbb{C}\otimes A)$ which maps into an invertible element in $KK_1(\mathbb{R}\ltimes(C_0(\mathbb{R})\otimes A), \mathbb{R}\ltimes(\mathbb{C}\otimes A))$.

But $\mathbb{R}\ltimes(C_0(\mathbb{R})\otimes A) \cong \mathcal{K}(L^2(\mathbb{R}))\otimes A$. Since KK_* is stable with respect to $\mathcal{K}(H)\otimes(-)$ (Morita equivalence), hence we get an invertible element in $KK_1(A, \mathbb{R} \rtimes A)$ inducing an isomorphism

$$K_*(\mathbb{R} \rtimes A) \cong K_{*+1}(A). \quad \square$$

Proposition. If G is discrete, then

$$K^*(G \rtimes A) \cong K^*_G(A). \quad \square$$

Remark. There is an invertible element in $KK_0(C_0(\mathbb{R}^2), \mathbb{C})$.

This can be generalized as follows.

Remark. If X is a compact Riemann manifold of negative curvature, then the universal cover $\widetilde{X}$ is diffeomorphic to $\mathbb{R}^n$. Let $G = \pi_1(X)$. It acts on $\mathbb{R}^n$ by deck transformations.

As before, there is an invertible element in $KK_n(C_0(\mathbb{R}^n), \mathbb{C})$.

It is G-invariant and remains invertible in $KK_n^G(C_0(\mathbb{R}^n), \mathbb{C})$.

How to construct such wonderful KK_*^G?

Naive idea: Take simply

$$KK^G_*(B,A) := \mathbf{Ab}^{\mathbb{Z}/2\mathbb{Z}}\left(K^G_*(B), K^G_*(A)\right).$$

Wrong! Why?

Example. D Dirac type operator on a smooth manifold X, Y another compact manifold,

$X \times Y \to Y$, a canonical bundle with fibre X.

D should give a map $K^*(X\times Y) \to K^*(Y)$, the <u>integration against the K-homology class</u> $[D]$.

One expects that once we view $[D]$ as an element of $KK_*(C(X), \mathbb{C})$ and the exterior products provides an element in the image of

$$KK_*(C(X), \mathbb{C}) \longrightarrow KK_*(C(X) \otimes C(Y), \mathbb{C} \otimes C(Y))$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$$

$$\qquad\qquad\qquad\qquad\qquad\qquad KK_*(C(X \times Y), C(Y))$$

Then applying the composition product in the diagram

$$\begin{array}{ccc} K^*(X\times Y) & & \\ \| & & \\ KK_*(\mathbb{C}, C(X\times Y)) & \xrightarrow{-\otimes[D]} & KK_*(\mathbb{C}, C(X\times Y)) \otimes KK_*(C(X\times Y), C(Y)) \\ & & \downarrow \\ & & KK_*(\mathbb{C}, C(Y)) \\ & & \| \\ & & K^*(Y). \end{array}$$

one should get the map

$$K^*(X\times Y) \longrightarrow K^*(Y).$$

However, although there is the Künneth map

$$K^*(X) \otimes K^*(Y) \longrightarrow K^*(X \times Y),$$

it is not an isomorphism in general, since $K^*(X \times Y)$ contains an additional piece $\mathrm{Tor}(K^*X, K^*Y)$, hence a map $K^*(X) \longrightarrow \mathbb{Z}$ defined as the pairing with a class $[D]$ does not directly induce a map $K^*(X \times Y) \longrightarrow K^*(Y)$.